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Exact Solution of Three-Dimensional Unsteady Stagnation Flow on a Heated Plate

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In this study three-dimensional unsteady stagnation flow and heat transfer impinging on a flat plate are investigated in a Cartesian coordinate system when the plate is moving with variable velocity and acceleration towards the main stream or away from it. An exact solution is obtained in this problem for the axisymmetric case. An external fluid, along z-direction, with strain rate a impinges on this flat plate and produces an unsteady three-dimensional flow in which the plate moves along z-direction with variable velocity and acceleration. A reduction of Navier–Stokes and energy equations is obtained by use of appropriate similarity transformations. Velocity, boundary-layer thickness, and surface stress-tensors along with temperature profiles are presented for different values of impinging fluid strain rate, different values of plate velocities, and Prandtl-number parameter for the case of steady state.

I. Introduction

EXACT solutions of Navier–Stokes and energy equations regarding the problem of stagnation-point flow and heat transfer in the vicinity of a flat plate or a cylinder are found in the literature for many cases. Fundamental studies in which flows are readily superposed and/or the axisymmetric case were considered include the following papers presented in the literature: two-dimensional stagnation-point flow [1]; three-dimensional stagnation-point flow [2]; and axisymmetric stagnation flow on a circular cylinder [3]. Further exact solutions to the Navier–Stokes equations are obtained by superposition of the uniform shear flow and/or stagnation flow on a body oscillating or rotating in its own plane or cylinder, with or without suction. The examples are: superposition of two-dimensional and three-dimensional stagnation-point flows [4]; superposition of stagnation-point flow on a flat plate oscillating in its own plate, and also consideration of the case where the plate is stationary and the stagnation stream is made to oscillate done [5]; heat Transfer in an axisymmetric stagnation flow on a cylinder [6]; nonsimilar axisymmetric stagnation flow on a moving cylinder [7]; unsteady viscous flow in the vicinity of an axisymmetric stagnation-point on a cylinder [8]; axisymmetric stagnation flow towards a moving plate [9]; axisymmetric stagnation-point flow impinging on a transversely oscillating plate with suction [10]; axisymmetric stagnation-point flow and heat transfer of a viscous fluid on a moving cylinder with time-dependent axial velocity and uniform transpiration [11] and also [12–14] can be mentioned.

In this study the three-dimensional unsteady viscous stagnation flow and heat transfer in the vicinity of an accelerated flat plate are investigated by solving Navier–Stokes equations in Cartesian coordinate system. The external fluid, along z-direction, with strain rate a impinges on this flat plate while the plate is moving with variable velocity and acceleration along z-direction. A self-similar solution for the Navier–Stokes equations and energy equation is derived in this problem. A reduction of these equations is obtained by use of these appropriate similarity transformations. The obtained ordinary differential equations are solved using numerical techniques. Velocity and pressure profiles, boundary-layer thickness and surface stress-tensors along with temperature profiles are presented for different values of impinging fluid strain rate, different plate velocities and Prandtl-number parameters for the steady state case.

II. Problem Formulation

Flow is considered in Cartesian coordinates (x, y, z) with corresponding velocity components (u, v, w). We consider the laminar unsteady incompressible flow and heat transfer of a viscous fluid in the neighborhood of stagnation-point on a moving flat plate located in the plane$z = 0$ at $t = 0$. An external fluid, along z-direction, with strain rate a impinges on this accelerated flat plate along z-direction and produces a three-dimensional flow on the plate. Obviously in the situation of moving plate the boundary-layer thickness along x-direction or y-direction changes in contrary to the case of fixed plate when they are with constant thickness. This accelerated plate, as an example, can be assumed as a solidification front which is moving with variable velocity along the z-axis. The unsteady Navier–Stokes and energy equations in Cartesian coordinates governing the flow and heat transfer are used in which p, ρ, ν, and α are the fluid pressure, density, kinematic viscosity, and thermal diffusivity.

III. Self-Similar Solution

The classical potential flow solution of the governing momentum equations is as follows:

$$U = ax \tag{1}$$

$$V = ay \tag{2}$$

$$W = -a(z - S(t)) \tag{3}$$

$$\begin{aligned} P_\infty &= P_0 - \frac{1}{2}\rho V^2 - \rho \int \frac{\partial V}{\partial t}\,\mathrm{d}r \\ &= P_0 - \frac{1}{2}\rho a^2[x^2 + y^2 + (z - S(t))^2] \\ &\quad - \rho \int \frac{\partial}{\partial t}[a\sqrt{x^2 + y^2 + (z - S(t))^2}]\,\mathrm{d}r \end{aligned} \tag{4}$$

Where $x = y$ (axisymmetric case),p_0 is stagnation pressure, r is the flow passage, V is total velocity, and$S(t)$ is the amount of plate

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displacement of plate in Z direction. A reduction of the Navier–Stokes equations is sought by the following coordinate separation in which the solution of the viscous problem inside the boundary layer is obtained by composing the inviscid and viscous parts of the velocity components [2,4,10]:

$$u = axf'(\eta) \tag{5}$$

$$v = ayf'(\eta) \tag{6}$$

$$w = -2\sqrt{a\upsilon}f(\eta) \tag{7}$$

$$\eta = \sqrt{\frac{a}{\upsilon}}(z - S(t)) \quad \text{and} \quad \zeta = (z - S(t)) \tag{8}$$

in which the terms involving $f(\eta)$ in Eqs. (5–8) comprise the Cartesian similarity form for unsteady stagnation-point flow and prime denotes differentiation with respect to η. Transformations (5–8) satisfy continuity equation automatically and their insertion into momentum equations yields an ordinary differential equation in terms of $f(\eta)$ and an expression for the pressure:

$$\begin{aligned} f''' + 2ff'' - (f')^2 + 1 = \tilde{\dot{S}}f'' \\ + \tilde{\dot{S}}\left[\frac{3\eta}{\tilde{x}^2 + \tilde{y}^2 + \eta^2} - \frac{1}{\sqrt{\tilde{x}^2 + \tilde{y}^2}}\tan^{-1}\frac{\eta}{\sqrt{\tilde{x}^2 + \tilde{y}^2}}\right] \\ + \int \frac{2\eta^2}{(\tilde{x}^2 + \tilde{y}^2 + \eta^2)2}\tilde{\dot{S}}\,\mathrm{d}\tilde{S}(t) \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{P} = -\left\{\int f'\tilde{\dot{S}}\,\mathrm{d}\eta + \frac{1}{2}f^2 + f'\right\} + \left\{\int \tilde{\dot{S}}\,\mathrm{d}\eta + \frac{1}{2}\eta^2 + 1\right\} \\ - \frac{1}{2}[\tilde{x}^2 + \tilde{y}^2 + \eta^2] + \eta\tilde{\dot{S}}Ln\left(\frac{\upsilon}{a}[\tilde{x}^2 + \tilde{y}^2 + \eta^2]\right) \\ + \eta\tilde{\dot{S}} - \sqrt{\tilde{x}^2 + \tilde{y}^2}\tilde{\dot{S}}\tan^{-1}\left[\frac{\eta}{\sqrt{\tilde{x}^2 + \tilde{y}^2}}\right] - \int \frac{\eta^2\tilde{\dot{S}}}{\tilde{x}^2 + \tilde{y}^2 + \eta^2}\,\mathrm{d}\tilde{S}_{(t)} \end{aligned} \tag{10}$$

where

$$\tilde{P} = \frac{p_{(x,y,z,t)}}{\rho a\upsilon} \qquad \tilde{S}(t) = \sqrt{\frac{a}{\upsilon}}S(t) \qquad \tilde{\dot{S}} = \dot{S}/\sqrt{a\upsilon}$$

$$\tilde{x} = \sqrt{\frac{a}{\upsilon}}x \qquad \tilde{y} = \sqrt{\frac{a}{\upsilon}}y$$

in which (.) denotes differentiation with respect to t, and quantities $\tilde{P}$, $\tilde{S}$, $\tilde{\dot{S}}$, $\tilde{x}$, and $\tilde{y}$ are nondimensional forms of quantities p, S, $\dot{S}$, x, and y, respectively. Relation (10) which represents pressure is obtained by integrating momentum equation in z-direction and by use of the potential flow solution (1–4) as boundary conditions. The boundary conditions for the differential Eq. (9) are

$$\eta = 0: \qquad f = 0, \qquad f' = 0 \tag{11}$$

$$\eta \to \infty: \qquad f' = 1 \tag{12}$$

Note that, when $\dot{S} = 0$ the case of Homman flow is obtained, [2], which is an axisymmetric case.

Making use of transformations (5–8), and

$$\theta = \frac{T(\eta) - T_\infty}{T_w - T_\infty} \tag{13}$$

the energy equation may be written as

$$\theta'' + Pr(\tilde{\dot{S}} + 2f)\theta' = 0 \tag{14}$$

With the boundary conditions as

$$\eta = 0: \qquad \theta = 1 \tag{15}$$

$$\eta \to \infty: \qquad \theta = 0 \tag{16}$$

Where $Pr = \nu/\alpha$, is Prandtl number and prime indicates differentiation with respect to η. Equations (14) and (9) are solved numerically using a shooting method trial and error and based on the Runge-Kutta algorithm and the results are presented for selected values of $\dot{S}$ and Pr in following sections.

IV. Presentation of Results

In this section the self-similar solution of Eqs. (14) and (9) along with the surface shear-stresses and pressure for selected values of plate velocity and Prandtl numbers are presented at specific distance$R$, measured from the stagnation point at $z = 0$ on the plate.

Figure 1 presents the u-component and or υ-component of velocity shown at $x = 4$ mm or $y = 4$ mm from the center of the jet. As the plate velocity towards the impinging flow increases up to 5 mm/s the thickness of the boundary layer increases while the parabolic form of the profile is kept but for the plate velocity greater than 5 mm/s the form of the profile changes though the increase of the thickness of the boundary layer slows down sharply. It is worth mentioning that the change of boundary-layer profile is in such a way that the increase of the velocity boundary-layer thickness is along with increase of the slope of these profiles which causes the shear-stress increase. The w-component of velocity inside the boundary layer has been depicted in Figs. 2 and 3. In Fig. 2, the large change of w-profile with increase of plate velocity is very much noticeable in which the profiles are more and more parabolic as the plate velocity increases. It can be seen in Fig. 3 that for plate velocity of 250 mm/s the increase of distance from center of jet in x or y direction causes not much of change in w-component of velocity and as the plate velocity decreases this change is less considerable.

Figure 4 shows the thermal boundary-layer profiles vs plate velocity and at different values of x or y and for different values of Prandtl number. Here, as Prandtl number increases the effect of increase of plate velocity on variation of profile of thermal boundary layer decreases. Figure 5 shows variation of thermal boundary-layer thickness. Here, the variation of boundary-layer thickness for a specific value of $x$or y and specific values of plate velocity and Prandtl number is presented. This figure shows that variation of thermal boundary-layer thickness for Prandtl numbers between 0.5 to around 20 is large and it levels off for Prandtl number bigger than that. It is interesting to note that this figure has an inflection point at

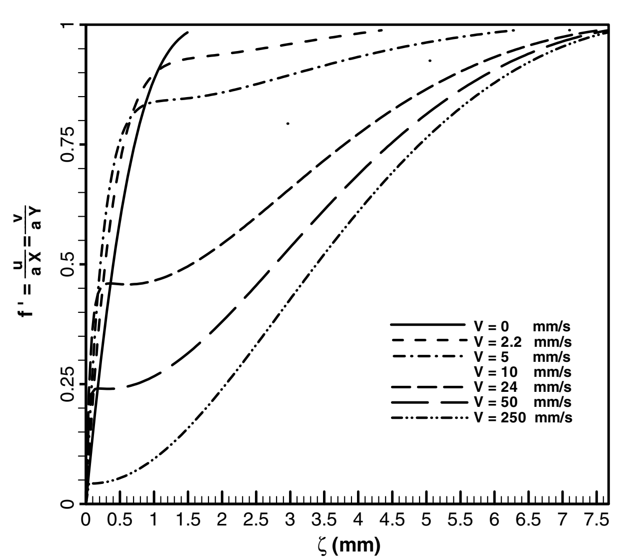

Fig. 1 ***u*- or *v*-component of velocity profile at *x* or *y* = 4 mm from *Z* axis for various plate velocity.**

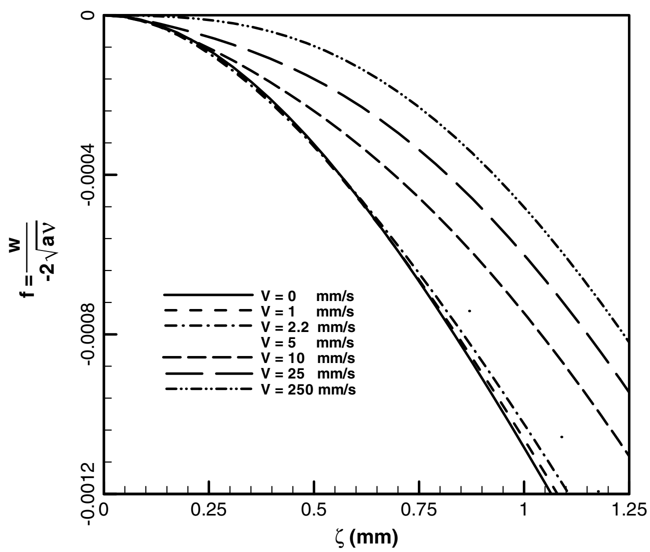

Fig. 2 ***w*-component of velocity profile for various plate velocity and for *x* or $y = 1$ mm.**

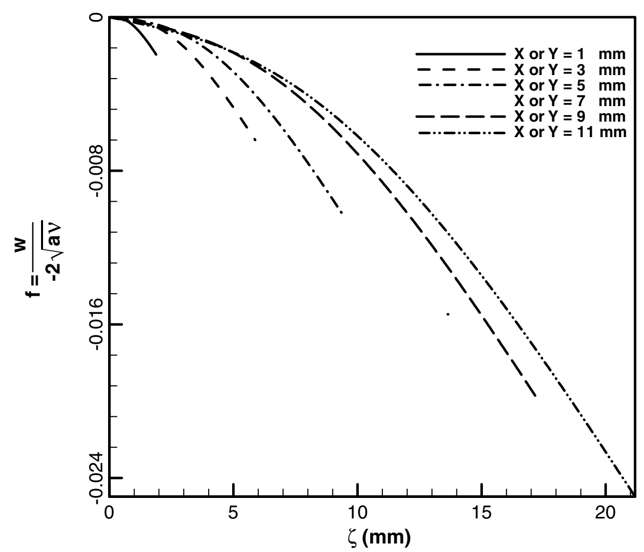

Fig. 3 ***w*-component of velocity profile for various *x* or *y* and plate velocity = 250 mm/s.**

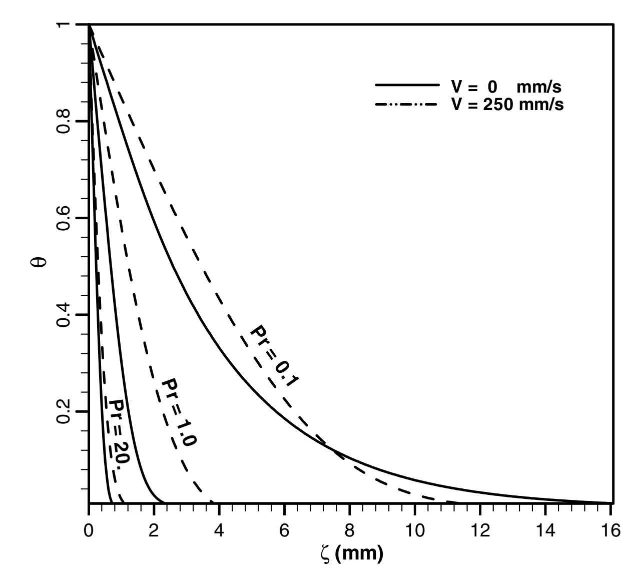

Fig. 4 **Temperature profile vs distance from plate for various Prandtl number and certain plate velocity, for temperature profiles maximum difference.**

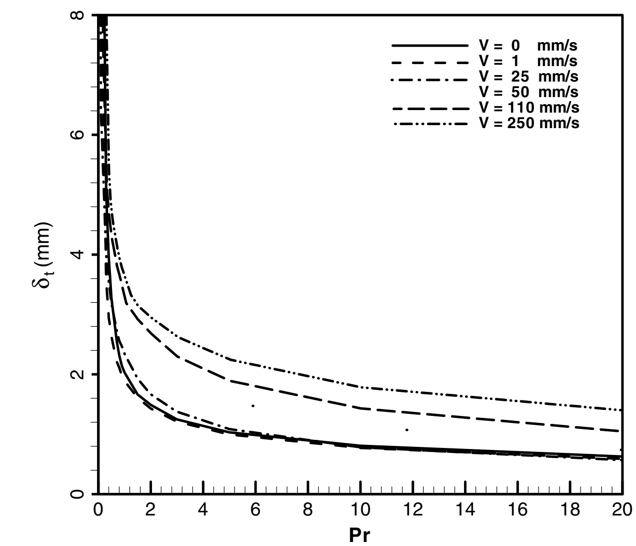

Fig. 5 **Thermal boundary-layer thickness vs *Pr* number and distance from *z*-axis = 4 mm and various plate velocity.**

around $Pr = 0.5$ in which variation of boundary-layer thickness increases as plate velocity increases but for $Pr > 0.5$ the boundary-layer thickness not only is not more than that if the velocity is zero but is even less.

V. Conclusions

Effects of moving plate along the main stream have been presented as an exact solution of the Navier–Stokes and energy equations for three-dimensional unsteady stagnation flow and heat transfer on a flat plate. Velocity components and temperature profiles have been presented for selected values of distance from the center of the jet (*x* or *y*), plate velocity, and Prandtl numbers for the steady state case. Increase in plate velocity produces interesting velocity profiles, boundary-layer thickness variations, and thermal thickness. It is interesting to note that there is an inflection point at $Pr = 0.5$ in which variation of boundary-layer thickness increases as plate velocity increases but for $Pr > 0.5$ the boundary-layer thickness not only is not more than if the plate velocity is zero but is even less. This is because the increase of Pr number decreases the thermal boundary layer but the increase of plate velocity increases this quantity and inflection point happens where the second-order and first-order terms in energy equation are balanced.

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References

[1] Hiemenz, K., "Die grenzchicht an einem in den gleichformingen Flussigkeitsstrom eingetauchten geraden KreisZylinder," *Dinglers Polytechnisches Journal*, Vol. 326, 1911, pp. 321–410.
[2] Homman, F. Z., "Der Einfluss Grosser Zahighkeit bei der Strmung um den Zylinder und um die Kugel," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 16, 1936, pp. 153–164. doi:10.1002/zamm.19360160304
[3] Wang, C. Y., "Axisymmetric Stagnation Flow on a Cylinder," *Quarterly of Applied Mathematics*, Vol. 32, 1974, pp. 207–213.
[4] Howarth, L., "The Boundary Layer in Three-Dimensional Flow. Part 2. The Flow Near Stagnation Point," *The Philosophical Magazine*, Vol. 42, 1951, pp. 1433–1440.
[5] Glauert, M. B., "The Laminar Boundary Layer on Oscillating Plates and Cylinders," *Journal of Fluid Mechanics*, Vol. 1, 1956, pp. 97–110. doi:10.1017/S002211205600007X
[6] Gorla, R. S. R., "Heat Transfer in an Axisymmetric Stagnation Flow on a Cylinder," *Applied Scientific Research*, Vol. 32, 1976, pp. 541–553. doi:10.1007/BF00385923

[7] Gorla, R. S. R., "Nonsimilar Axisymmetric Stagnation Flow on a Moving Cylinder," *International Journal of Engineering Science*, Vol. 16, 1978, pp. 392–400.

[8] Gorla, R. S. R., "Unsteady Viscous Flow in the Vicinity of an Axisymmetric Stagnation-Point on a Cylinder," *International Science and Technology*, Vol. 17, 1979, pp. 87–93.

[9] Wang, C. Y., "Axisymmetric Stagnation Flow Towards a Moving Plate," *American Institute of Chemical Engineers Journal*, Vol. 19, No. 5, 1973, p. 961.

[10] Weidman, P. D., and Mahalingam, S., "Axisymmetric Stagnation-Point Flow Impinging on a Transversely Oscillating Plate With Suction," *Journal of Engineering Mathematics*, Vol. 31, 1997, pp. 305–318.
doi:10.1023/A:1004211515780

[11] Saleh, R., and Rahimi, A. B., "Axisymmetric Stagnation-Point Flow and Heat Transfer of a Viscous Fluid on a Moving Cylinder with Time-Dependent Axial Velocity and Uniform Transpiration," *Journal of Fluids Engineering*, Vol. 126, No. 6, 2004, pp. 997–1005.
doi:10.1115/1.1845556

[12] Rahimi, A. B., and Saleh, R., "Axisymmetric Stagnation-Point Flow and Heat Transfer of a Viscous Fluid on a Rotating Cylinder with Time-Dependent Angular Velocity and Uniform Transpiration," *Journal of Fluids Engineering*, Vol. 129, No. 1, 2007, pp. 107–115.

[13] Shokrgozar Abbasi, A., and Rahimi, A. B., "Non-Axisymmetric Three-Dimensional Stagnation-Point Flow and Heat Transfer on a Flat Plate," *Journal of Fluids Engineering*, Vol. 131, No. 7, 2009, pp. 074501.1–074501.5.

[14] Shokrgozar Abbasi, A., and Rahimi, A. B., "Three-Dimensional Stagnation-Point Flow and Heat Transfer on a Flat Plate with Transpiration," *Journal of Thermophysics and Heat Transfer*, Vol. 23, No. 3, July–Sept. 2009, pp. 513–521.
doi:10.2514/1.41529